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PAPER-DCQ10

A Thermodynamic Reading of the Adapted Morse Landscape

Finite Partition Functions and Formal Thimble-Sector Bookkeeping in the DCQ Correspondence

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Abstract

This paper develops a thermodynamic reading of the adapted Morse landscape arising in the Discrete–Continuous–Quantum correspondence. The starting point is the compact six-dimensional phase-orbit manifold

$$\mathcal{N} \simeq (\mathbb{C}P^1)^3,$$

together with an adapted Morse function

$$f : \mathcal{N} \to \mathbb{R}$$

whose prescribed local minima include the 64 finite phase-sector states inherited from the six-bit configuration space

$$\mathcal{H}_6 = \{\pm 1\}^6.$$

The main purpose of the paper is deliberately limited. We do not claim a first-principles derivation of microscopic thermodynamics from a physical Hamiltonian, nor do we assume that DCQ2 has already supplied a rigorous Picard–Lefschetz convergence theorem. Instead, we show that once the adapted Morse function is interpreted as an effective energy landscape, the compact real integral

$$\mathcal{Z}(\beta) = \int_{\mathcal{N}} e^{-\beta f} \, d\mu_{\mathcal{N}}$$

defines a finite equilibrium partition function. The formal thimble language is used only as a semiclassical sector bookkeeping device for organizing contributions from critical points.

Two consequences are emphasized. First, in the low-temperature regime, the 64 prescribed minima contribute a discrete ground-sector degeneracy term

$$k_B \ln 64,$$

after separating non-universal local fluctuation-volume factors. Second, a simplified two-level critical-sector model exhibits a Schottky-type heat-capacity peak, illustrating how excited Morse sectors may become relevant at intermediate temperature scales.

Thus the paper isolates a mathematically stable thermodynamic layer attached to the DCQ Morse landscape while leaving stronger dynamical, gravitational, and phenomenological interpretations to later work.

Paper-specific keywords: adapted Morse landscape, finite partition function, effective energy landscape, ground-sector degeneracy, formal thimble-sector bookkeeping, Schottky-type anomaly.

Geometric keywords: $(\mathbb{C}P^1)^3$, compact symplectic manifold, Fubini–Study measure, Morse function, critical sectors, phase-orbit geometry.

Framework keywords: Discrete–Continuous–Quantum correspondence, six-bit binary configuration space, finite phase sector, formal thimble ansatz, structural ground-sector entropy, finite thermal readout.

Contents

1 Introduction

The earlier papers in the DCQ series established a geometric correspondence linking a finite six-bit configuration space to continuous Grassmannian and phase-orbit geometry. In the revised formulation, the structural chain begins with

$$\mathcal{H}_6 = \{\pm 1\}^6,$$

passes through the phase-encoded embedding into

$$\mathrm{Gr}(3,6),$$

and admits a continuous phase-orbit completion

$$\mathcal{N} \simeq (\mathbb{C}P^1)^3.$$

The 64 elements of $\mathcal{H}_6$ appear inside $\mathcal{N}$ as a finite μ_4^3-labelled subset.

Paper DCQ2 formulated an adapted Morse-potential framework on this compact six-dimensional phase-orbit manifold. In particular, one may choose a smooth Morse function

$$f : \mathcal{N} \to \mathbb{R}$$

whose prescribed local minima include the 64 embedded finite phase-sector states. This raises a natural question:

Can the adapted Morse landscape on $\mathcal{N}$ be given a coherent thermodynamic reading?

The present paper answers this question in a limited but stable sense. The Morse function f is interpreted as an effective energy landscape, and the compact real integral

$$\mathcal{Z}(\beta) = \int_{\mathcal{N}} e^{-\beta f}\, d\mu_{\mathcal{N}}$$

is taken as the associated equilibrium partition function. Since $\mathcal{N}$ is compact and f is smooth, this integral is finite for all

$$\beta > 0.$$

Thus the basic thermodynamic construction does not require a rigorous Picard–Lefschetz convergence theorem.

The formal thimble language still has a useful role. It provides a semiclassical bookkeeping framework for organizing the low- temperature expansion by critical sectors:

$$\mathcal{Z}(\beta) \sim \sum_{p \in \mathrm{Crit}(f)} A_p(\beta) e^{-\beta f(p)}.$$

In the present paper, this expression is treated as a formal asymptotic organization, not as a completed analytic theorem about complex integration cycles.

Remark DCQ10-1.1 (Main change of emphasis)**.** Earlier versions of this line of thought sometimes spoke as if the Morse–thimble decomposition itself were already a strict analytic input. The present paper uses a safer formulation:

$$\text{finite real partition function} + \text{formal thimble-sector bookkeeping}.$$

The thermodynamic reading is therefore based on the compact Morse landscape, not on an assumed proof of thimble convergence.

The paper has four aims:

(1) to define a thermodynamic partition function from the adapted Morse landscape on $\mathcal{N}$;

(2) to derive the associated free energy, internal energy, entropy, and heat capacity;

(3) to isolate the $k_B \ln 64$ contribution as a discrete ground-sector degeneracy term;

(4) to show, through a two-level critical-sector model, how Schottky-type heat-capacity peaks arise from excited Morse sectors.

Remark DCQ10-1.2 (Scope of the paper)**.** This is a bridge paper. It does not derive a microscopic Hamiltonian, spacetime dynamics, gravitational thermodynamics, or particle physics. It records a controlled thermodynamic reading of the adapted Morse landscape already present in the DCQ geometric framework.

The paper is organized as follows. Section 2 recalls the DCQ geometric input. Section 3 interprets the Morse function as an effective energy landscape. Section 4 defines the finite real partition function and its formal critical-sector expansion. Section 5 derives the thermodynamic quantities. Section 7 clarifies the meaning of the $k_B \ln 64$ term. Section 8 presents a two-level Schottky-type model. Section 10 concludes.

2 Geometric Background from the DCQ Framework

We recall only the structural ingredients required for the thermodynamic reading.

2.1 Discrete finite phase sector

The starting space is the six-bit configuration space

$$\mathcal{H}_6 = \{\pm 1\}^6,$$

with

$$|\mathcal{H}_6| = 2^6 = 64.$$

In DCQ1, the six bits are grouped into three pairs. Each pair determines one of four fourth-root phase values, and the three pairs define a finite phase sector

$$\mathcal{H}_6 \longrightarrow \mu_4^3 \subset U(1)^3.$$

2.2 Phase-orbit completion

DCQ2 replaces the finite phase set by its projective continuous completion. The resulting compact phase-orbit manifold is

$$\mathcal{N} \simeq (\mathbb{C}P^1)^3.$$

It is a real six-dimensional compact Khler and symplectic manifold equipped with the product Fubini–Study structure.

The finite phase-sector states appear as the subset

$$\mathcal{N}_{\text{disc}} = \{([1:\zeta_1],[1:\zeta_2],[1:\zeta_3]) \mid \zeta_i \in \mu_4\} \subset (\mathbb{C}P^1)^3,$$

with

$$|\mathcal{N}_{\text{disc}}| = 4^3 = 64.$$

2.3 Adapted Morse potential

The Morse-theoretic layer of DCQ2 may be summarized as follows.

Definition DCQ10-2.1 (Adapted Morse potential)**.** An adapted Morse potential on $\mathcal{N}$ is a smooth Morse function

$$f : \mathcal{N} \to \mathbb{R}$$

such that:

(i) the 64 finite phase-sector points in $\mathcal{N}_{\rm disc}$ are prescribed nondegenerate local minima of f;

(ii) all remaining critical points, if present, are nondegenerate;

(iii) after shifting f by a constant, the prescribed minima have value

$$f = 0.$$

Remark DCQ10-2.2 (Non-uniqueness)**.** The adapted Morse function is not unique. Different choices of f may have different higher critical-point structures. The present paper uses only the existence of an adapted Morse landscape with 64 prescribed local minima.

Proposition DCQ10-2.3 (Finite ground-sector structure)**.** *Under the adapted Morse-potential framework, the finite phase-sector states form a* 64*-fold prescribed ground sector after the normalization*

$$f|_{\mathcal{N}_{\rm disc}} = 0.$$

Proof. By definition, each point of $\mathcal{N}_{\rm disc}$ is a prescribed local minimum of f. Since

$$|\mathcal{N}_{\rm disc}| = 64,$$

and the function is shifted so that these minima have value zero, they form a 64-fold prescribed zero-energy sector under the effective energy reading introduced below. □

3 The Morse Function as an Effective Energy Landscape

We now introduce the thermodynamic interpretation.

Definition DCQ10-3.1 (Effective energy landscape)**.** Let

$$f : \mathcal{N} \to \mathbb{R}$$

be an adapted Morse potential. Choose an energy scale

$$\varepsilon_0 > 0.$$

The associated effective energy function is

$$E(p) = \varepsilon_0 f(p), \qquad p \in \mathcal{N}.$$

Throughout most of the paper we set

$$\varepsilon_0 = 1$$

to simplify notation, so that $E = f$.

Remark DCQ10-3.2 (Interpretive status)**.** The effective energy $E = \varepsilon_0 f$ is not claimed to be a fundamental physical Hamiltonian. It is a thermodynamic reading of the adapted Morse landscape. A later dynamical theory may derive or modify such an energy function.

The key point is that a Morse function already stratifies $\mathcal{N}$ into:

$$\text{minima}, \qquad \text{saddles}, \qquad \text{higher critical sectors.}$$

This is exactly the type of structure that can support equilibrium statistical mechanics once a positive Boltzmann weighting is chosen.

Principle DCQ10-3.3 (Thermodynamic reading principle)**.** The adapted Morse potential f may be read as an effective energy landscape for a finite equilibrium thermal model on the compact phase-orbit manifold

$$\mathcal{N} \simeq (\mathbb{C}P^1)^3.$$

Under this reading, the 64 finite phase-sector states form the prescribed ground sector, while higher critical points represent excited geometric sectors.

4 Finite Partition Function and Formal Critical-Sector Expansion

The safest thermodynamic object is the real compact partition function.

4.1 The finite real partition function

Let

$$\beta = (k_B T)^{-1}$$

be the inverse temperature. Let $d\mu_{\mathcal{N}}$ be a positive finite measure on $\mathcal{N}$, for example the normalized Liouville or Khler volume measure.

Definition DCQ10-4.1 (Finite Morse partition function)**.** The finite Morse partition function associated with f is

$$\mathcal{Z}(\beta) = \int_{\mathcal{N}} e^{-\beta f(p)}\, d\mu_{\mathcal{N}}(p).$$

Proposition DCQ10-4.2 (Finiteness)**.** *For every $\beta > 0$, the partition function*

$$\mathcal{Z}(\beta) = \int_{\mathcal{N}} e^{-\beta f}\, d\mu_{\mathcal{N}}$$

is finite.

Proof. The manifold $\mathcal{N} \simeq (\mathbb{C}P^1)^3$ is compact, and f is smooth. Hence f is bounded below and above. Therefore

$$e^{-\beta f}$$

is a bounded positive function on a finite-measure space. Its integral is finite. □

Remark DCQ10-4.3 (Why no thimble convergence is needed here)**.** The finiteness of $\mathcal{Z}(\beta)$ follows directly from compactness and smoothness. It does not require a complexification of $\mathcal{N}$, a Picard–Lefschetz cycle construction, or a proof of thimble convergence.

4.2 Formal critical-sector bookkeeping

Although the real partition function is sufficient for thermodynamics, the Morse structure suggests an asymptotic expansion organized by critical points.

Let

$$\mathrm{Crit}(f)$$

denote the finite set of critical points of f. In the low-temperature regime, Laplace asymptotics formally gives

$$\mathcal{Z}(\beta) \sim \sum_{p \in \mathrm{Crit}(f)} A_p(\beta) e^{-\beta f(p)},$$

where $A_p(\beta)$ is a local fluctuation factor determined by the Hessian and local measure near p.

For a nondegenerate local minimum p in real dimension 6, the standard Gaussian contribution has the form

$$A_p(\beta) \sim g_p \, \beta^{-3},$$

where

$$g_p > 0$$

depends on the Hessian determinant and measure normalization.

Principle DCQ10-4.4 (Formal thimble-sector bookkeeping)**.** The critical-sector expansion

$$\mathcal{Z}(\beta) \sim \sum_{p \in \mathrm{Crit}(f)} A_p(\beta) e^{-\beta f(p)}$$

may be interpreted as a formal thimble-sector bookkeeping device. It organizes contributions by critical sectors, but it is not used here as a strict Picard–Lefschetz convergence theorem.

Remark DCQ10-4.5 (Relation to DCQ2)**.** This is the thermodynamic counterpart of the formal Morse–thimble ansatz in DCQ2. The present paper only requires the real finite partition function. The thimble terminology helps organize asymptotic sectors but does not carry the burden of analytic convergence.

5 Thermodynamic Quantities

Given the finite partition function $\mathcal{Z}(\beta)$, define the usual equilibrium quantities:

$$F(\beta) = -\frac{1}{\beta} \ln \mathcal{Z}(\beta),$$

$$U(\beta) = -\frac{\partial}{\partial \beta} \ln \mathcal{Z}(\beta),$$

$$S(\beta) = k_B \left(\ln \mathcal{Z}(\beta) + \beta U(\beta) \right),$$

and

$$C(\beta) = -k_B \beta^2 \frac{\partial U}{\partial \beta}.$$

5.1 Expectation-value form

Since

$$\mathcal{Z}(\beta) = \int_{\mathcal{N}} e^{-\beta f}\, d\mu_{\mathcal{N}},$$

the normalized thermal expectation of an observable A is

$$\langle A \rangle_\beta = \frac{1}{\mathcal{Z}(\beta)} \int_{\mathcal{N}} A(p) e^{-\beta f(p)}\, d\mu_{\mathcal{N}}(p).$$

Proposition DCQ10-5.1 (Internal energy and heat capacity)**.** *The internal energy is*

$$U(\beta) = \langle f \rangle_\beta,$$

and the heat capacity is

$$C(\beta) = k_B \beta^2 \,\mathrm{Var}_\beta(f),$$

where

$$\mathrm{Var}_\beta(f) = \langle f^2 \rangle_\beta - \langle f \rangle_\beta^2.$$

Proof. Differentiating

$$\ln \mathcal{Z}(\beta)$$

gives

$$-\frac{\partial}{\partial \beta} \ln \mathcal{Z}(\beta) = \frac{\int_{\mathcal{N}} f e^{-\beta f} d\mu_{\mathcal{N}}}{\int_{\mathcal{N}} e^{-\beta f} d\mu_{\mathcal{N}}} = \langle f \rangle_\beta.$$

Differentiating once more,

$$\frac{\partial U}{\partial \beta} = -\,\mathrm{Var}_\beta(f).$$

Therefore

$$C(\beta) = -k_B \beta^2 \frac{\partial U}{\partial \beta} = k_B \beta^2 \,\mathrm{Var}_\beta(f).$$

□

Remark DCQ10-5.2 (Positivity)**.** The heat capacity is nonnegative in this finite equilibrium model:

$$C(\beta) \geq 0.$$

This follows immediately from the variance formula.

6 Low- and High-Temperature Regimes

We now examine the two basic regimes.

6.1 Low-temperature regime

Assume that the 64 prescribed minima have:

$$f(p) = 0, \qquad p \in \mathcal{N}_{\mathrm{disc}}.$$

Assume also, for simplicity, that their local Gaussian prefactors are equal:

$$g_p = g_0.$$

Then the formal low-temperature asymptotic contribution from the minima is

$$\mathcal{Z}(\beta) \sim 64\, g_0\, \beta^{-3}$$

in real dimension 6, up to exponentially suppressed contributions from higher critical values.

Hence

$$\ln \mathcal{Z}(\beta) \sim \ln 64 + \ln g_0 - 3 \ln \beta.$$

Proposition DCQ10-6.1 (Low-temperature asymptotics)**.** *Under the equal-prefactor approximation for the* 64 *minima,*

$$U(\beta) \sim \frac{3}{\beta},$$

and

$$S(\beta) \sim k_B \left(\ln 64 + \ln g_0 - 3 \ln \beta + 3\right),$$

up to exponentially suppressed higher-sector corrections.

Proof. From

$$\ln \mathcal{Z}(\beta) \sim \ln 64 + \ln g_0 - 3 \ln \beta,$$

we obtain

$$U(\beta) = -\partial_\beta \ln \mathcal{Z}(\beta) \sim \frac{3}{\beta}.$$

Then

$$S(\beta) = k_B(\ln \mathcal{Z} + \beta U) \sim k_B \left(\ln 64 + \ln g_0 - 3 \ln \beta + 3\right).$$

□

Remark DCQ10-6.2 (Meaning of the $k_B \ln 64$ term)**.** The term

$$k_B \ln 64$$

is the discrete ground-sector degeneracy contribution. The additional terms

$$\ln g_0 - 3 \ln \beta + 3$$

come from local continuous fluctuation volumes and measure normalization. They are not universal discrete entropy terms.

Remark DCQ10-6.3 (No physical third-law claim)**.** Because the model uses a continuous classical phase-space integral near each minimum, the low-temperature entropy contains fluctuation-volume terms such as $-3 \ln \beta$. Therefore the expression should not be read as a literal third-law quantum thermodynamic entropy. The robust structural contribution is the discrete degeneracy factor 64, hence $k_B \ln 64$.

6.2 High-temperature regime

Assume the measure is normalized:

$$\int_{\mathcal{N}} d\mu_{\mathcal{N}} = 1.$$

For $\beta \to 0$, expand:

$$e^{-\beta f} = 1 - \beta f + \frac{\beta^2}{2} f^2 + O(\beta^3).$$

Thus

$$\mathcal{Z}(\beta) = 1 - \beta \langle f \rangle_0 + \frac{\beta^2}{2} \langle f^2 \rangle_0 + O(\beta^3),$$

where $\langle \cdot \rangle_0$ denotes uniform averaging over $\mathcal{N}$.

Taking the logarithm gives

$$\ln \mathcal{Z}(\beta) = -\beta \langle f \rangle_0 + \frac{\beta^2}{2} \operatorname{Var}_0(f) + O(\beta^3).$$

Therefore

$$U(\beta) = \langle f \rangle_0 - \beta \operatorname{Var}_0(f) + O(\beta^2),$$

and

$$C(\beta) = k_B \beta^2 \operatorname{Var}_0(f) + O(\beta^3).$$

Remark DCQ10-6.4 (High-temperature interpretation)**.** At high temperature, the full compact geometry of $\mathcal{N}$ becomes relevant. The thermodynamics is then controlled by global averages over the phase-orbit manifold rather than by the 64 discrete minima alone.

7 The Meaning of the $k_B \ln 64$ Contribution

The DCQ framework contains several finite counts. In this paper, the relevant one is

$$|\mathcal{H}_6| = 64.$$

This count appears because the adapted Morse landscape has 64 prescribed finite phase-sector minima.

Definition DCQ10-7.1 (Structural ground-sector entropy)**.** The structural ground-sector entropy associated with the prescribed 64-fold finite Morse-minimum sector is

$$S_{\mathrm{gs}} := k_B \ln 64.$$

Remark DCQ10-7.2 (Distinction from $\ln 24$)**.** The quantity

$$\ln 64$$

belongs to the discrete finite phase-sector and to the 64-fold Morse ground sector. By contrast,

$$\ln 24$$

belongs to the pure Bose–Fermi readout carrier

$$\mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4)$$

discussed in DCQ1 and DCQ3. These are different structural layers and should not be identified.

Principle DCQ10-7.3 (Layer separation of entropy-like counts)**.** The DCQ programme contains distinct logarithmic counts:

$$\ln 64 \quad \text{for the finite six-bit/Morse-minimum sector,}$$

and

$$\ln 24 \quad \text{for the pure Bose–Fermi readout sector.}$$

The present paper concerns the first count.

Remark DCQ10-7.4 (No black-hole entropy derivation)**.** The structural ground-sector term $k_B \ln 64$ is not a derivation of black-hole entropy. It is a degeneracy contribution arising from the 64 prescribed Morse minima of the DCQ phase-orbit landscape.

8 A Two-Level Critical-Sector Model and Schottky-Type Anomaly

We now present a simple model showing how an excited critical sector can produce a heat-capacity peak.

8.1 Discrete two-level approximation

For maximal clarity, first ignore continuous fluctuation prefactors and consider a two-level discrete approximation:

$$\mathcal{Z}_{2\mathrm{lev}}(\beta) = g_0 + g_1 e^{-\beta\Delta},$$

where:

$$g_0 > 0$$

is the ground-sector degeneracy,

$$g_1 > 0$$

is the excited-sector degeneracy, and

$$\Delta > 0$$

is the energy gap.

In the DCQ application, one may take

$$g_0 = 64$$

for the prescribed ground sector, while g_1 represents an effective count of excited critical-sector contributions.

The internal energy is

$$U(\beta) = \frac{g_1 \Delta e^{-\beta\Delta}}{g_0 + g_1 e^{-\beta\Delta}}.$$

The heat capacity is

$$C(\beta) = k_B(\beta\Delta)^2 \frac{g_0 g_1 e^{-\beta\Delta}}{(g_0 + g_1 e^{-\beta\Delta})^2}.$$

Proposition DCQ10-8.1 (Schottky-type heat-capacity peak)**.** *The two-level partition function*

$$\mathcal{Z}_{2\mathrm{lev}}(\beta) = g_0 + g_1 e^{-\beta\Delta}$$

exhibits a Schottky-type heat-capacity peak at an intermediate temperature scale satisfying

$$\beta\Delta = O(1),$$

up to degeneracy-dependent logarithmic shifts.

Proof. At very low temperature,

$$\beta\Delta \gg 1,$$

the excited term is exponentially suppressed, so $C(\beta) \to 0$. At very high temperature,

$$\beta\Delta \ll 1,$$

the factor $(\beta\Delta)^2$ suppresses $C(\beta)$. Therefore $C(\beta)$ must peak in the crossover regime, where

$$g_1 e^{-\beta\Delta}$$

becomes comparable to g_0. This occurs when

$$\beta\Delta \sim \ln(g_1/g_0),$$

when the logarithm is relevant, and generically at

$$\beta\Delta = O(1).$$

□

8.2 Including fluctuation prefactors

A more faithful Morse-sector approximation includes local Gaussian factors:

$$\mathcal{Z}_{\text{crit}}(\beta) = A\beta^{-a} + B\beta^{-b}e^{-\beta\Delta}.$$

Here $A, B > 0$ encode effective critical-sector weights, and a, b encode local fluctuation dimensions.

The qualitative mechanism remains unchanged:

$$A\beta^{-a} \sim B\beta^{-b}e^{-\beta\Delta}$$

marks the crossover between ground-sector and excited-sector dominance. This crossover produces a heat-capacity peak.

Remark DCQ10-8.2 (Role of the toy model)**.** The two-level model is not a full computation of the critical network of $\mathcal{N}$. It isolates the minimal mechanism by which excited Morse sectors produce a Schottky-type anomaly.

9 Limitations and Future Directions

The present paper is intentionally limited.

It establishes:

adapted Morse landscape $\Rightarrow$ finite partition function $\Rightarrow$ thermodynamic quantities $\Rightarrow$ ground-sector degenerac

It does not establish:

(i) a microscopic physical Hamiltonian;

(ii) a rigorous Picard–Lefschetz convergence theorem;

(iii) a unique contour prescription for complex thimble integrals;

(iv) a derivation of black-hole thermodynamics;

(v) a physical prediction for a real material or cosmological system.

Remark DCQ10-9.1 (Correct role of thimble language)**.** The thimble language in this paper is a formal sector organization, not the foundation of the partition function. The partition function itself is the real compact integral over $\mathcal{N}$.

Future work may develop:

- explicit adapted Morse functions on $(\mathbb{C}P^1)^3$;
- numerical computation of their full critical networks;
- a rigorous complexified Picard–Lefschetz companion theory;
- dynamical models whose Hamiltonians reduce to an adapted Morse landscape;
- connections between DCQ thermodynamic readout and FBT coarse-grained entropy structures.

10 Conclusion

This paper has given a thermodynamic reading of the adapted Morse landscape in the DCQ framework.

The central construction is the finite partition function

$$\mathcal{Z}(\beta) = \int_{\mathcal{N}} e^{-\beta f} \, d\mu_{\mathcal{N}}, \qquad \mathcal{N} \simeq (\mathbb{C}P^1)^3.$$

Because $\mathcal{N}$ is compact and f is smooth, this partition function is well-defined and finite for all positive inverse temperatures. The formal thimble-sector expansion is then used only as semiclassical bookkeeping:

$$\mathcal{Z}(\beta) \sim \sum_{p \in \mathrm{Crit}(f)} A_p(\beta) e^{-\beta f(p)}.$$

The 64 finite phase-sector states inherited from

$$\mathcal{H}_6 = \{\pm 1\}^6$$

appear as prescribed minima of the adapted Morse landscape. Therefore the low-temperature structure contains the discrete ground-sector degeneracy contribution

$$S_{\mathrm{gs}} = k_B \ln 64,$$

separated from non-universal local fluctuation-volume terms.

Finally, a two-level critical-sector model shows how excited Morse sectors can produce Schottky-type heat-capacity anomalies at intermediate temperature scales.

The value of DCQ4 is therefore organizational and structural. It does not derive full physical thermodynamics from first principles, but it shows that the Morse landscape naturally supports a finite equilibrium thermal reading compatible with the revised DCQ1–DCQ3 foundations.

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AI Tool Usage: The author used large language models during manuscript preparation for brainstorming, structural revision, and exposition refinement. All central ideas, mathematical constructions, claims, and final scientific responsibility remain with the author.

A On the Positive Thermal Reading

The main text uses the positive real partition function

$$\mathcal{Z}(\beta) = \int_{\mathcal{N}} e^{-\beta f}\, d\mu_{\mathcal{N}}.$$

This appendix explains its relation to the formal thimble language.

In an oscillatory complexified setting, one might write a formal expression

$$Z_{\mathrm{osc}} = \sum_{p \in \mathrm{Crit}(f)} n_p \int_{\mathcal{J}_p} e^{-S/\hbar}\, d\mu,$$

where n_p are intersection numbers, $\mathcal{J}_p$ are thimbles, and S may be complex.

This is not the object used as the thermodynamic partition function in the present paper. For equilibrium thermodynamics, the relevant object must be real and positive. Therefore the present paper works with

$$\mathcal{Z}(\beta) = \int_{\mathcal{N}} e^{-\beta f}\, d\mu_{\mathcal{N}},$$

and treats the thimble-sector language as an asymptotic organization of critical contributions.

Remark DCQ10-A.1. This avoids depending on a strict analytic construction of complex thimbles. The compact real partition function is already sufficient for the thermodynamic layer developed in DCQ4.

B Layer Separation: $\ln 64$ and $\ln 24$

The broader DCQ framework contains at least two important logarithmic counts.

First,

$$|\mathcal{H}_6| = 64.$$

This gives

$$\ln 64,$$

which is the logarithm of the full finite six-bit configuration count. In DCQ4, it appears as the ground-sector degeneracy contribution associated with the 64 prescribed Morse minima.

Second, DCQ1 and DCQ3 distinguish a pure Bose–Fermi readout carrier

$$\mathcal{R}_{\mathrm{BF}} = \mathrm{Sym}^3(\mathbb{C}^4) \oplus \Lambda^3(\mathbb{C}^4),$$

with

$$\dim \mathcal{R}_{\mathrm{BF}} = 24.$$

This gives the separate readout count

$$\ln 24.$$

Count	**Object**	**Meaning**
$\ln 64$	$\mathcal{H}_6$ / 64 Morse minima	finite phase-sector or ground-sector degeneracy
$\ln 24$	$\mathcal{R}_{\mathrm{BF}}$	pure Bose–Fermi readout entropy

Remark DCQ10-B.1. DCQ4 concerns $\ln 64$, not $\ln 24$. The two counts belong to different layers and should not be merged.

C Minimal Citation Form for Later Papers

Later papers may cite DCQ4 in the following compact form:

> DCQ4 gives a thermodynamic reading of the adapted Morse landscape on
>
> $$\mathcal{N} \simeq (\mathbb{C}P^1)^3.$$
>
> Given an adapted Morse function $f : \mathcal{N} \to \mathbb{R}$ whose prescribed minima include the 64 finite phase-sector states, DCQ4 defines the finite real partition function
>
> $$\mathcal{Z}(\beta) = \int_{\mathcal{N}} e^{-\beta f} \, d\mu_{\mathcal{N}}.$$
>
> The formal thimble-sector language is used only as semiclassical bookkeeping, not as a strict convergence theorem. The 64 minima contribute a structural ground-sector degeneracy term $k_B \ln 64$, distinct from the separate $\ln 24$ pure Bose–Fermi readout count.

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